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EXPONENTIAL GENERATING FUNCTIONS AND COMPLEXITY OF LIE VARIETIES

BY

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ABSTRACT

Suppose that $\mathbb{V}$ is a variety of Lie algebras, and let $c_n(\mathbb{V})$ be the dimension of the linear span of all multilinear words on n distinct letters in the free algebra $F(\mathbb{V}, X)$ of the variety $\mathbb{V}$. We consider an exponential generating function

$$\mathcal{C}(\mathbb{V}, z) = \sum_{n=1}^{\infty} \frac{c_n(\mathbb{V})}{n!} z^n,$$

called the complexity function. The complexity function is an entire function of a complex variable provided the variety of Lie algebras is nontrivial. In this paper we introduce the notion of complexity for Lie varieties in terms of the growth of complexity functions; also we describe what the complexity means for the codimension growth of the variety. Our main goal is to specify the complexity of a product of two Lie varieties in terms of the complexities of multiplicands. The main observation here is that $\mathcal{C}(\mathbb{M}\mathbb{V}, z)$ behaves like a composition of three functions $\mathcal{C}(\mathbb{M}, z)$, $\exp(z)$, and $\mathcal{C}(\mathbb{V}, z)$.

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1. Codimension growth of varieties of Lie algebras

The structure of the present paper is as follows. In Section 1 we present some facts on the codimension growth of Lie varieties. Section 2 contains some necessary facts on the growth of entire functions. In Section 3 we introduce the notion of complexity for Lie varieties and discuss its connection with the codimension growth (Theorem 3.5). In Section 4 we study double complexity functions; they happen to be a good instrument. In Section 5 we prove Theorem 5.1 on the complexity of the product of two Lie varieties; this is our main result.

Let H be a Lie algebra over a field K. Suppose that $F = F(X)$ is the free Lie algebra with the countable set of free generators $X = \{x_i | i \in \mathbb{N}\}$. Then $0 \neq f \in F$ is said **identity** (or **identical relation**) in H, iff for any $a_i \in H$, $i \in \mathbb{N}$ and any homomorphism ϕ: $F \to H$, $\phi(x_i) = a_i$, $i \in \mathbb{N}$ we have $\phi(f) = 0$; also in this case we say that H is a **PI-algebra**. Suppose that $I \subset F$; then the class of all Lie algebras $\mathbb{V}$, satisfying identities I, is said **variety**. The set of all identities $\mathbb{V}(X)$, which are true in a given variety $\mathbb{V}$, is an ideal in $F(X)$, so we can consider $F(\mathbb{V}, X) = F(X)/\mathbb{V}(X)$, which is called **free algebra of the variety** $\mathbb{V}$. It has the property that $\forall H \in \mathbb{V}$, $\forall a_i \in H$, $i \in \mathbb{N}$, $\exists! \phi$: $F(\mathbb{V}, X) \to H$: $\phi(x_i) = a_i$, $i \in \mathbb{N}$. A variety is called **nontrivial** if it satisfies some nonzero identity. For the theory of varieties of Lie algebras see the monograph [1].

Let $P_n(\mathbb{V}) \subset F(\mathbb{V}, X)$ be the set of multilinear elements on letters $x_1, \dots, x_n$. Namely, $P_n(\mathbb{V})$ is the linear span of all monomials on $x_1, \dots, x_n$ (more exactly, their images in $F(\mathbb{V}, X)$) such that each letter x_i, $i = 1, \dots, n$ enters monomials exactly one time. For the variety $\mathbb{V}$ we have the function of the **codimension growth**

$$c_n(\mathbb{V}) = c_n(F(\mathbb{V}, X), X) = \dim_K P_n(\mathbb{V}), \quad n \in \mathbb{N}.$$

It is evident that for other variables $x_{i_1}, \dots, x_{i_n} \in X$ we will get the same number. The behavior of the sequence $c_n(\mathbb{V})$ is an important characteristic of the variety $\mathbb{V}$. The same observations hold also for varieties of associative algebras.

Recall the definition of the polynilpotent variety of Lie algebras $\mathbb{N}_{s_q} \cdots \mathbb{N}_{s_2} \mathbb{N}_{s_1}$, corresponding to a tuple $(s_q, \dots, s_2, s_1)$; it consists of all Lie algebras H, such that there exists a chain of ideals $0 = H_{q+1} \subset H_q \subset \cdots \subset H_1 = H$, $H_i/H_{i+1} \in \mathbb{N}_{s_i}$. If $s_q = \cdots = s_2 = s_1 = 1$ then one has the variety $\mathbb{A}^q$ of solvable Lie algebras of step q. Lie brackets are considered to be leftnormed, that is $[a_1, \dots, a_n] = [\dots[a_1, a_2], \dots, a_n]$.

In [15] we suggested the scale to measure an overexponential growth of Lie algebras; in [16] this scale was refined. In terms of this scale the growth of polynilpotent varieties of Lie algebras was described. Define by iteration

$$\ln^{(0)} x = x,\ \ln^{(s+1)} x = \ln(\ln^{(s)} x);\quad \exp^{(0)} x = x,\ \exp^{(s+1)} x = \exp(\exp^{(s)} x),$$
$$s = 0, 1, 2, \ldots .$$

We consider the series of functions with two real parameters α, β:

$$(1)\qquad \Psi^q_{\alpha,\beta}(n) = \begin{cases} (n!)^{(\alpha-1)/\alpha}\beta^{n/\alpha}, & \alpha \geq 1,\ \beta > 0; \qquad q = 2; \\ \dfrac{n! \cdot (\beta/\alpha)^{n/\alpha}}{(\ln^{(q-2)} n)^{n/\alpha}}, & \alpha > 0,\ \beta > 0; \qquad q = 3, 4, \ldots . \end{cases}$$

THEOREM 1.1 ([16]): *Let $\mathbb{V} = \mathbb{N}_{s_q} \cdots \mathbb{N}_{s_1}$, $q \geq 2$ be the polynilpotent variety. Then there exists an infinitesimal such that*

$$c_n(\mathbb{V}) = \Psi^q_{s_1, s_2+o(1)}(n) = \begin{cases} (n!)^{(s_1-1)/s_1}(s_2 + o(1))^{n/s_1}; & q = 2; \\ \dfrac{n!}{(\ln^{(q-2)} n)^{n/s_1}}\left(\dfrac{s_2 + o(1)}{s_1}\right)^{n/s_1}; & q = 3, 4, \ldots . \end{cases}$$

By the following result we see that the scale above is rather complete.

THEOREM 1.2 ([16]): *Let $\mathbb{V}$ be a variety of Lie algebras satisfying some non-trivial identity of degree $m > 3$. Then there exists an infinitesimal (depending only on m) such that*

$$c_n(\mathbb{V}) \leq \frac{n!}{(\ln^{(m-3)} n)^n}(1 + o(1))^n.$$

This is an analogue of Regev's theorem [19], which asserts that for associative PI-algebras c_n is bounded by some exponent.

2. Growth of entire functions

For two functions $f(x)$, $g(x)$ we write $f(x) \overset{a}{\leq} g(x)$ iff $\exists R$: $f(x) \leq g(x), x \geq R$; we call it the **asymptotic inequality**. Recall the notion of **order** [20], [3], [10] for an entire function $f(z)$ of the complex variable. Denote $\mathrm{M}_f(r) = \max_{|z|=r} |f(z)|$, $r \in \mathbb{R}^+$;

$$\operatorname{ord} f = \inf\{\rho | \mathrm{M}_f(r) \overset{a}{\leq} \exp(r^\rho)\} = \varlimsup_{r\to\infty} \frac{\ln\ln \mathrm{M}_f(r)}{\ln r}.$$

Also, **type** is defined, provided that the order $\rho = \operatorname{ord} f$ is known [20], [3]:

$$\operatorname{typ} f = \inf\{\sigma | \mathrm{M}_f(r) \overset{a}{\leq} \exp(\sigma r^\rho)\} = \varlimsup_{r\to\infty} \frac{\ln \mathrm{M}_f(r)}{r^\rho}.$$

By analogy we define the **order** and **type** of the **level** q, $q = 0, 1, 2, \dots$ as (the second provided that $\operatorname{ord}_q f = \rho$)

$$\operatorname{ord}_q f = \inf\{\rho | \mathrm{M}_f(r) \overset{\mathrm{a}}{\le} \exp^{(q)}(r^\rho)\} = \overline{\lim_{r\to\infty}} \frac{\ln^{(q+1)} \mathrm{M}_f(r)}{\ln r};$$

$$\operatorname{typ}_q f = \inf\{\sigma | \mathrm{M}_f(r) \overset{\mathrm{a}}{\le} \exp^{(q)}(\sigma r^\rho)\} = \overline{\lim_{r\to\infty}} \frac{\ln^{(q)} \mathrm{M}_f(r)}{r^\rho}.$$

We need two facts, the first of which is the classical result. Both assertions connect the growth of an entire function with the asymptotic behavior of its coefficients.

THEOREM 2.1 ([3, 3.2.3], [10]): *Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be an entire function of order ρ and type σ. Then*

$$\overline{\lim_{n\to\infty}} n^{1/\rho} \sqrt[n]{|a_n|} = (\sigma e \rho)^{1/\rho}.$$

THEOREM 2.2 ([16]): *Suppose that $f(z) = \sum_{n=0}^\infty a_n z^n$ is an entire function. Then for any fixed numbers $q \in \mathbb{N}, q \ge 2$; $\rho > 0$,*

$$\overline{\lim_{r\to\infty}} \frac{\ln^{(q)} \mathrm{M}_f(r)}{r^\rho} = \overline{\lim_{n\to\infty}} \ln^{(q-1)} n \, |a_n|^{\rho/n}.$$

These two theorems have a weaker version.

THEOREM 2.3: *Suppose that $f(z) = \sum_{n=0}^\infty a_n z^n$ is an entire function. Then the order of level $q, q \in \mathbb{N}$, is equal to*

$$\operatorname{ord}_q f(z) = \overline{\lim_{n\to\infty}} \frac{\ln^{(q)} n}{\ln \sqrt[n]{1/|a_n|}}.$$

For the ordinary order ($q = 1$) this is the classical result [10]. If $q \ge 2$ then the result follows from the estimates obtained in the proof of Theorem 2.2; see [16].

We shall make use of one technical lemma.

LEMMA 2.1: *Let $q \in \mathbb{N}$, $\rho > 0$, $\sigma > 0$ be fixed numbers.*

(1) *The following formulas give an entire function with order ρ, type σ, and level q.*

$$f(z) = \sum_{n=N}^{\infty} a_n z^n,$$

$$a_n = \begin{cases} \left(\dfrac{e\sigma\rho}{n}\right)^{n/\rho}, & q = 1; \\ \left(\dfrac{\sigma}{\ln^{(q-1)} n}\right)^{n/\rho}, & q \ge 2 \qquad (\text{where } \ln^{(q-1)} N > 1). \end{cases}$$

(2) *For these functions* $\mathrm{M}_f(r) = \exp^{(q)}((\sigma + o(1))r^\rho)$, $r \in \mathbb{R}^+$.

(3) *Under integration* $g(z) = \int_0^z f(z)dz$ *the behavior of the module remains the same,* $\mathrm{M}_g(r) = \exp^{(q)}((\sigma + o(1))r^\rho)$, $r \in \mathbb{R}^+$.

Proof: The first assertion follows directly from Theorems 2.1 and 2.2 (for $q = 1$ this is well known [10]).

The upper bound for $\mathrm{M}_f(r)$ in the second claim follows from the fact that $f(z)$ has the given order and type. It remains to obtain the lower bound. First, we consider $q = 1$,

$$\mathrm{M}_f(r) = \sum_{n=N}^{\infty} a_n r^n \geq \gamma(n) = a_n r^n = \left(\frac{e\sigma\rho}{n}\right)^{n/\rho} r^n, \quad r \in \mathbb{R}^+.$$

By taking $n_* = \sigma\rho r^\rho$ one has $\gamma(n_*) = \exp(\sigma r^\rho)$; this is the desired lower bound (the whole part for n_* does not seriously change matters). If $q \geq 2$ then

$$\mathrm{M}_f(r) \geq \gamma(n) = a_n r^n = \left(\frac{\sigma}{\ln^{(q-1)} n}\right)^{n/\rho} r^n, \quad r \in \mathbb{R}^+.$$

Computation shows that for any $\theta > 1$, by taking $n_* = \exp^{(q-1)}(\sigma/\theta r^\rho)$ we obtain the estimate

$$\begin{aligned} \mathrm{M}_f(r) \geq \gamma(n_*) &= \exp\left(\frac{\ln\theta}{\rho}\exp^{(q-1)}(\sigma/\theta r^\rho)\right) \\ &\overset{\mathrm{a}}{\geq} \exp^{(q)}(\sigma_0/\theta r^\rho), \qquad \text{for any } \sigma_0 < \sigma. \end{aligned} \tag{2}$$

This is the desired lower bound since $\theta > 1$, $\sigma_0 < \sigma$ were taken arbitrarily.

For assertion (3), the upper bound follows from the fact that the order, type, and level of an entire function do not change after integration. For the ordinary order and type this is well known [3]. For higher levels, see [16]. Let $g(z) = \int_0^z f(z)dz = \sum_{n=1}^{\infty} b_n z^n$; then $b_n = a_{n-1}/n$ and, for $q \geq 2$,

$$b_n = \frac{1}{n}\left(\frac{\sigma}{\ln^{(q-1)}(n-1)}\right)^{(n-1)/\rho} \overset{\mathrm{a}}{\geq} \left(\frac{\widetilde{\sigma}}{\ln^{(q-1)} n}\right)^{n/\rho}, \quad \text{for any } \widetilde{\sigma} < \sigma.$$

An analogous estimate is valid for $q = 1$. Now we can apply the estimates obtained above ((2) for $q \geq 2$) to obtain the desired lower bounds. ∎

3. Complexity of Lie varieties

First, recall the notion of **complexity** (or **PI-degree**) for varieties of associative algebras due to V. N. Latyshev. Suppose that $\mathbb{V}$ is a variety of associative algebras. Then $n(\mathbb{V})$ is the maximal n such that the matrix ring $M_n(K)$ belongs to $\mathbb{V}$. For a PI-algebra A denote by $\operatorname{var} A$ the variety determined by all identical relations of A; then its complexity is defined as $n(A) = n(\operatorname{var} A)$. By the theorem of Amitsur [9] the complexity of a PI-algebra is always a finite number.

THEOREM 3.1 ([4], [5], [6]): *For any associative PI-algebras A, B the following inequalities hold:*

$$n(A) \cdot n(B) \leq n(A \otimes B) \leq 2n(A) \cdot n(B).$$

Now let us turn to varieties of Lie algebras. We will introduce the notion of complexity for Lie varieties in terms of the growth of complexity functions. Let $\mathbb{V}$ be a variety of Lie algebras; then the **complexity function** is defined as [17], [18]

$$\mathcal{C}(\mathbb{V}, z) = \sum_{n=1}^{\infty} \frac{c_n(\mathbb{V})}{n!} z^n, \quad z \in \mathbb{C}.$$

This notion enabled Yu. P. Razmyslov to reformulate in a nice way the upper bound for an arbitrary nontrivial variety of Lie algebras due to Grishkov [9]:

THEOREM 3.2 ([17], [18]): *Let $\mathbb{V}$ be a nontrivial variety of Lie algebras. Then $\mathcal{C}(\mathbb{V}, z)$ is an entire function of the complex variable.*

An important step in the proof of Theorem 1.1 is an estimate on the growth of the corresponding complexity function. In case of complexity functions we have, of course, $\mathrm{M}_f(r) = f(r)$, $r \in \mathbb{R}^+$.

THEOREM 3.3 ([16]): *Let $\mathbb{V} = \mathbb{N}_{s_q} \cdots \mathbb{N}_{s_1}$, $q \geq 2$ be the polynilpotent variety and $f(z) = \mathcal{C}(\mathbb{V}, z)$ be its complexity function. Then*

$$\lim_{r \to \infty} \frac{\ln^{(q-1)} \mathrm{M}_f(r)}{r^{s_1}} = \frac{s_2}{s_1}.$$

COROLLARY 3.1 ([16]): *Let $\mathbb{V} = \mathbb{N}_{s_q} \cdots \mathbb{N}_{s_1}$, $q \geq 2$ be the polynilpotent variety and $f(z) = \mathcal{C}(\mathbb{V}, z)$ be its complexity function. Then*

$$\operatorname{ord}_{q-1} f(z) = s_1, \qquad \operatorname{typ}_{q-1} f(z) = s_2/s_1.$$

Also, Theorem 1.2 was derived from estimates for the complexity function. Denote $\sum_{n=0}^{\infty} a_n z^n \prec \sum_{n=0}^{\infty} b_n z^n$ iff $a_n \leq b_n$, $n \geq 0$. The following result is an improvement of the bound [18].

THEOREM 3.4 ([16]): *Let $\mathbb{V}$ be a variety of Lie algebras satisfying some nontrivial identity of degree $m > 3$. Then there exists an entire function $r(z)$ (depending only on m) such that*

$$\mathcal{C}(\mathbb{V},z) \prec r(z), \quad \operatorname{ord}_{m-2} r(z) = \operatorname{typ}_{m-2} r(z) = 1.$$

Now let us define the **complexity numbers** of a Lie variety $\mathbb{V}$ in terms of the growth of its complexity function $f(z) = \mathcal{C}(\mathbb{V},z)$.

$$\begin{aligned}
\operatorname{Comp}^1 \mathbb{V} &= \min\{q \in \mathbb{N} \mid \exists \rho\colon \mathrm{M}_f(r) \overset{\mathrm{a}}{\leq} \exp^{(q-1)}(r^\rho)\};\\
\operatorname{Comp}^2 \mathbb{V} &= \inf\{\rho \mid \mathrm{M}_f(r) \overset{\mathrm{a}}{\leq} \exp^{(q-1)}(r^\rho)\} = \operatorname{ord}_{q-1} \mathcal{C}(\mathbb{V},z);\\
\operatorname{Comp}^3 \mathbb{V} &= \inf\{\sigma \mid \mathrm{M}_f(r) \overset{\mathrm{a}}{\leq} \exp^{(q-1)}(\sigma r^\rho)\} = \operatorname{typ}_{q-1} \mathcal{C}(\mathbb{V},z).
\end{aligned}$$

Let us comment on these numbers. By Theorem 3.4, $q = \operatorname{Comp}^1 \mathbb{V} \in \mathbb{N}$ is always a finite number for a nontrivial variety; it will also be referred to as the **level** of $\mathbb{V}$. By construction $\rho = \operatorname{Comp}^2 \mathbb{V} \in [0,\infty)$ is also finite. In case $\rho = \operatorname{Comp}^2 \mathbb{V} > 0$ we also consider $\operatorname{Comp}^3 \mathbb{V}$. Now by the **complexity** of $\mathbb{V}$ we mean the tuple $\operatorname{Comp} \mathbb{V} = (\operatorname{Comp}^1 \mathbb{V}, \operatorname{Comp}^2 \mathbb{V}, \operatorname{Comp}^3 \mathbb{V})$. We say that $\operatorname{Comp} \mathbb{V}$ is nontrivial iff the second and third entries are finite and nonzero. To evaluate from below we also introduce numbers

$$\begin{aligned}
\underline{\operatorname{Comp}}^1 \mathbb{V} &= \max\{p \in \mathbb{N} \mid \exists \alpha > 0\colon \mathrm{M}_f(r) \overset{\mathrm{a}}{\geq} \exp^{(p-1)}(r^\alpha)\};\\
\underline{\operatorname{Comp}}^2 \mathbb{V} &= \sup\{\alpha > 0 \mid \mathrm{M}_f(r) \overset{\mathrm{a}}{\geq} \exp^{(p-1)}(r^\alpha)\}, \text{ where } p = \underline{\operatorname{Comp}}^1 \mathbb{V};\\
\underline{\operatorname{Comp}}^3 \mathbb{V} &= \sup\{\beta \geq 0 \mid \mathrm{M}_f(r) \overset{\mathrm{a}}{\geq} \exp^{(p-1)}(\beta r^\alpha)\}, \text{ for } \alpha = \underline{\operatorname{Comp}}^2 \mathbb{V} < \infty;\\
\underline{\operatorname{Comp}} \mathbb{V} &= (\underline{\operatorname{Comp}}^1 \mathbb{V}, \underline{\operatorname{Comp}}^2 \mathbb{V}, \underline{\operatorname{Comp}}^3 \mathbb{V}).
\end{aligned}$$

We shall compare tuples (q,ρ,σ) lexicographically from the left and use sign $\leq$. In these notations Theorems 3.3 and 3.4 imply

COROLLARY 3.2: *Let $\mathbb{V} = \mathbb{N}_{s_q} \cdots \mathbb{N}_{s_1}$. Then*

$$\operatorname{Comp} \mathbb{V} = \underline{\operatorname{Comp}} \mathbb{V} = \begin{cases} (q, s_1, 1/s_1), & q = 1;\\ (q, s_1, s_2/s_1), & q \geq 2. \end{cases}$$

Proof: We should only add for $q = 1$ that $c_n(\mathbb{N}_s) = (n-1)!$, $1 \leq n \leq s$, and $\mathcal{C}(\mathbb{N}_s, z) = \sum_{n=1}^{s} z^n/n$. ∎

COROLLARY 3.3: *Suppose that $\mathbb{V}$ is a variety of Lie algebras satisfying a nontrivial identity of degree m. Then* $\operatorname{Comp} \mathbb{V} \leq (m-1,1,1)$.

Also, let us consider the series of functions $\Psi_\alpha^q(n)$, $q = 2,3,\dots$ with one real parameter α, which yields a rougher scale than (1):

$$\Psi_\alpha^q(n) = \begin{cases} \big(n!\big)^{(\alpha-1)/\alpha}, & \alpha > 1, \quad q = 2; \\ \dfrac{n!}{(\ln^{(q-2)} n)^{n/\alpha}}, & \alpha > 0, \quad q = 3,4,\dots. \end{cases} \tag{3}$$

Let $f(n)$ be a function of a natural argument, and suppose that $\tau_\gamma(n)$ is a function of a natural argument, which is continuous and increasing with respect to the parameter γ (e.g. (3), or (1)). We will compare $f(n)$ with $\tau_\gamma(n)$ in the following way. Let the number $c \in \mathbb{R}$ be fixed. Then we introduce a convenient notation:

$$f(n) \sim^c \tau_c(n) \iff \inf\{\gamma \mid f(n) \overset{\mathrm{a}}{\leq} \tau_\gamma(n)\} = c.$$

In [13] Mishchenko introduced **exponents** for a variety of Lie algebras $\mathbb{V}$; further, they are also studied in [14]:

$$\operatorname{Exp} \mathbb{V} = \varlimsup_{n\to\infty} \sqrt[n]{c_n(\mathbb{V})}, \quad \underline{\operatorname{Exp}}\, \mathbb{V} = \varliminf_{n\to\infty} \sqrt[n]{c_n(\mathbb{V})}.$$

Now we can describe what these complexity numbers mean for the codimension growth.

THEOREM 3.5: *Suppose that $\mathbb{V}$ is a nontrivial variety of Lie algebras with* $\operatorname{Comp}^1 \mathbb{V} = q$.

(1) *If $q = 1$ then:*
  - (a) *the complexity function is a polynomial $\mathcal{C}(\mathbb{V}, z) = \sum_{n=1}^{\rho} c_n(\mathbb{V})/n! z^n$, and* $\operatorname{Comp}^2 \mathbb{V} = \underline{\operatorname{Comp}}^2 \mathbb{V} = \rho$, $\operatorname{Comp}^3 \mathbb{V} = \underline{\operatorname{Comp}}^3 \mathbb{V} = c_\rho(\mathbb{V})/\rho!$;
  - (b) *$\mathbb{V}$ is nilpotent $\mathbb{V} \subset \mathbb{N}_\rho$, $\mathbb{V} \not\subset \mathbb{N}_{\rho-1}$.*

(2) *If $q \geq 2$ then the situation is described by comparison with the scales in equations (3) and (1):*
  - (a) $\operatorname{Comp}^2 \mathbb{V} = \rho$ *iff* $c_n(\mathbb{V}) \sim^\rho \Psi_\rho^q(n)$;
  - (b) $\operatorname{Comp}^3 \mathbb{V} = \sigma$ *iff* $c_n(\mathbb{V}) \sim^\sigma \Psi_{\rho,\sigma\cdot\rho}^q(n)$, *provided that* $\operatorname{Comp}^2 \mathbb{V} = \rho$;
  - (c) *if $q = 2$, $\rho = 1$ then* $\operatorname{Comp}^3 \mathbb{V} = \operatorname{Exp} \mathbb{V}$.

Proof: If $q = 1$ then the complexity function is bounded by some polynomial and, by the Cauchy estimates, it coincides with some polynomial [20]; $\mathcal{C}(\mathbb{V}, z) = \sum_{n=1}^{\rho} c_n(\mathbb{V})/n! z^n$. Evidently, $c_{\rho+1}(\mathbb{V}) = 0$, $c_\rho(\mathbb{V}) \neq 0$ implies that $\mathbb{V} \subset \mathbb{N}_\rho$, $\mathbb{V} \not\subset \mathbb{N}_{\rho-1}$.

Let us prove case (2b). Recall that in our case $f(z) = \mathcal{C}(\mathbb{V}, z) = \sum_{n=1}^{\infty} a_n z^n$, $a_n = c_n(\mathbb{V})/n!$ If $q \geq 3$, then we use Theorem 2.2 and observe that the following conditions are equivalent:

$$\varlimsup_{n\to\infty} \ln^{(q-2)} n|a_n|^{\rho/n} = \varlimsup_{r\to\infty} \frac{\ln^{(q-1)} \mathrm{M}_f(r)}{r^\rho} = \mathrm{typ}_{q-1}\, \mathcal{C}(\mathbb{V},z) = \sigma;$$
$$\ln^{(q-2)} n|a_n|^{\rho/n} \sim^\sigma \sigma; \qquad a_n \sim^\sigma \frac{\sigma^{n/\rho}}{(\ln^{(q-2)} n)^{n/\rho}};$$
$$c_n(\mathbb{V}) = a_n n! \sim^\sigma \frac{n!\cdot \sigma^{n/\rho}}{(\ln^{(q-2)} n)^{n/\rho}} = \Psi^q_{\rho,\sigma\cdot\rho}(n).$$

If $q = 2$, then we apply Theorem 2.1 and substitute by the Stirling formula $n^n \approx n! e^n/\sqrt{2\pi n}$:

$$\varlimsup_{n\to\infty} n^{1/\rho}|a_n|^{1/n} = (\sigma e\rho)^{1/\rho}, \quad c_n(\mathbb{V}) = a_n n! \sim^\sigma n^{-n/\rho}(\sigma e\rho)^{n/\rho} n!;$$
$$c_n(\mathbb{V}) \sim^\sigma (n!)^{1-1/\rho}(\sigma\rho)^{n/\rho} = \Psi^2_{\rho,\sigma\cdot\rho}(n). \tag{4}$$

Now case (2c) follows from (4).

For case (2a), we apply Theorem 2.3 and proceed as above:

$$\varlimsup_{n\to\infty} \frac{\ln^{(q-1)} n}{\ln \sqrt[n]{1/|a_n|}} = \mathrm{ord}_{q-1}\, \mathcal{C}(\mathbb{V},z) = \mathrm{Comp}^2\, \mathbb{V} = \rho; \qquad \frac{\ln^{(q-1)} n}{\ln \sqrt[n]{1/|a_n|}} \sim^\rho \rho;$$
$$|a_n| \sim^\rho \frac{1}{(\ln^{(q-2)} n)^{n/\rho}}; \quad c_n(\mathbb{V}) = n!\cdot|a_n| \sim^\rho \frac{n!}{(\ln^{(q-2)} n)^{n/\rho}}.$$

If $q = 2$ then we also use an easy corollary from the Stirling formula $n^n = (n!)^{1+o(1)}$. ∎

Remark: It is not possible to interpret in the same way $\underline{\mathrm{Comp}}\, \mathbb{V}$.

Remark: It is also possible to consider the polynomial growth of c_n; this is the subcase of case (2c). Let us describe the situation for $\mathrm{char}\, K = 0$. Varieties of Lie algebras with the polynomial codimension growth were specified in [2], [11]. The condition that c_n grow polynomially is equivalent to $\mathrm{Exp}\, \mathbb{V} = 1$; the latter in our notions can be written as $\mathrm{Comp}\, \mathbb{V} = (2,1,1)$. Indeed, if c_n grow less than any exponent (i.e. $\mathrm{Exp}\, \mathbb{V} = 1$) then this is a polynomial [12]. Moreover, the recent result of S. P. Mishchenko says that $\mathrm{Exp}\, \mathbb{V} < 2$ implies the polynomial growth [13].

4. Double complexity functions

We shall consider complexity functions in a more general context. Suppose that we are given an algebra (groupoid), which is generated by $X = \{x_i | i \in \mathbb{N}\}$. Let A be a subspace in this algebra (subset in the groupoid).

For any set of distinct elements $\widetilde{X} = \{x_{i_1}, \ldots, x_{i_n}\} \subset X$, by $P_n(A, \widetilde{X})$ we denote the linear span (subset) of all multilinear elements of degree n on $\widetilde{X}$ in A. Suppose that the dimension of this linear span (number of elements) $c_n(A, \widetilde{X})$ does not depend on the choice of $\widetilde{X}$, but depends only on n. In this case we write $c_n(A) = c_n(A, \widetilde{X})$; also in this case we say that, for A, we have **defined the complexity function with respect to** X; the latter is written as

$$\mathcal{C}(A, z) = \sum_{n=1}^{\infty} \frac{c_n(A)}{n!} z^n, \quad z \in \mathbb{C}$$

(where the sum is taken from $n = 0$, $c_0 = 1$ for associative algebras and groupoids with unity). Remark that A need not necessarily consist of multilinear elements. The complexity function is one example of *exponential generating functions* [7].

Also, we shall consider double complexity functions. Suppose that we have an algebra (groupoid), which is generated by two sets $X = \{x_i | i \in \mathbb{N}\}$, $Y = \{y_i | i \in \mathbb{N}\}$. Let A be a subspace (subset in the groupoid). For any sets of distinct elements $\widetilde{X} = \{x_{i_1}, \ldots, x_{i_n}\} \subset X$, $\widetilde{Y} = \{y_{i_1}, \ldots, y_{i_m}\} \subset Y$ we denote by $P_{nm}(A, \widetilde{X}, \widetilde{Y})$ the linear span (subset) of all multilinear elements of degrees n on $\widetilde{X}$ and m on $\widetilde{Y}$ in A. Suppose that the dimension of this linear span (number of elements) $c_{nm}(A, \widetilde{X}, \widetilde{Y})$ does not depend on the choice of $\widetilde{X}, \widetilde{Y}$, but depends only on n, m. In this case we write $c_{nm}(A) = c_{nm}(A, \widetilde{X}, \widetilde{Y})$; also, in this case we say that, for A, we have **defined the double complexity function with respect to** X and Y:

$$\mathcal{C}(A, x, y) = \sum_{n=1, m=1}^{\infty} \frac{c_{nm}(A)}{n! m!} x^n y^m, \qquad x, y \in \mathbb{C}$$

(where the sum is taken from $n = 0, m = 0$ $c_{00} = 1$ for associative algebras and groupoids with unity). We also hope that the variables $x, y \in \mathbb{C}$ will not be mixed with $x_i \in X$, $y_j \in Y$. Of course, it is possible to consider triple complexity functions, etc. When it is clear what complexity function we consider, we omit variables and write $\mathcal{C}(A)$.

The following two lemmas are the double versions of [7], [18], [17], [15], [16].

LEMMA 4.1: *Let Γ be a groupoid (an algebra) as above with a multiplication denoted by $*$. Suppose that for subsets (subspaces) $A \subset \Gamma$, $B \subset \Gamma$ the double* *complexity functions are defined. Suppose also that all multilinear elements among $a * b$, $a \in A$, $b \in B$ of the set $A * B \subset \Gamma$ are distinct (all multilinear elements among $\{a_i * b_j | i \in I, j \in J\}$ are linearly independent provided that we take linearly independent sets $\{a_i | i \in I\} \subset A$ and $\{b_j | j \in J\} \subset B$). Then $A * B$ has a double complexity function, and $\mathcal{C}(A * B) = \mathcal{C}(A)\mathcal{C}(B)$.*

Proof: We observe that

$$c_{nm}(A * B) = \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} c_{ij}(A) c_{n-i,m-j}(B),$$

where the binomial coefficients correspond to all divisions of sets $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_m\}$ into two parts. This yields the assertion for the series. ∎

LEMMA 4.2: *Let $L = F(\mathbb{V}, X)$ be a Lie algebra generated by sets $X = \{x_i | i \in \mathbb{N}\}$, $Y = \{y_i | i \in \mathbb{N}\}$ and the double complexity function is defined with respect to these sets. Then its universal enveloping algebra $U(L)$ also has the double complexity function with respect to X, Y. Moreover, $\mathcal{C}(U(L)) = \exp(\mathcal{C}(L))$.*

Proof: Let us make use of the standard construction of the universal enveloping algebra:

$$U(L) \cong T(L) / \operatorname{Id}\{a \otimes b - b \otimes a - [a, b] | a, b \in L\},$$
$$T(L) = \bigoplus_{m=0}^{\infty} T^m, \quad T^m = \underbrace{L \otimes \cdots \otimes L}_{m}.$$

By Lemma 4.1 we obtain $\mathcal{C}(T^m) = \mathcal{C}(L)^m$. Let $\{y_j | j \in J\}$ be a homogeneous basis for L. We count multilinear elements only, hence the number of multilinear elements among $y_{j_1} \otimes \cdots \otimes y_{j_m} \in T^m$ is $m!$ times bigger than the number of the corresponding ordered elements in the basis of $U(L)$. As a result, one has

$$\mathcal{C}(U(L)) = \sum_{m=0}^{\infty} \frac{\mathcal{C}(T^m)}{m!} = \sum_{m=0}^{\infty} \frac{\mathcal{C}(L)^m}{m!} = \exp(\mathcal{C}(L)). \qquad ∎$$

LEMMA 4.3: *Let L be a Lie algebra generated by $X = \{x_i | i \in \mathbb{N}\}$ (by $X = \{x_i | i \in \mathbb{N}\}$ and $Y = \{y_i | i \in \mathbb{N}\}$). Suppose that a homogeneous set $\Xi \subset L$ has the (double) complexity function $\mathcal{C}(\Xi)$ with respect to X (X and Y). Also suppose that Ξ freely generates $H = F(\mathbb{V}, \Xi)$, the free algebra of a multihomogeneous variety $\mathbb{V}$; and $\mathbb{V}$ has the ordinary complexity function $\mathcal{C}(\mathbb{V}, z)$. Then H with respect to X (X and Y) has the following (double) complexity function:*

$$\mathcal{C}(H) = \mathcal{C}(\mathbb{V}, \mathcal{C}(\Xi)).$$

Proof: Suppose that

$$C(\mathbb{V}, z) = \sum_{n=1}^{\infty} \frac{c_n}{n!} z^n. \tag{5}$$

If we consider the set of ordered n-tuples $\Xi_n = (\xi_1, \ldots, \xi_n | \xi_i \in \Xi)$, then by Lemma 4.1, $C(\Xi_n) = C(\Xi)^n$ ($\Xi_n, n \in \mathbb{N}$ correspond to the elements of the free monoid generated by Ξ). The complexity function enumerates multilinear elements only, in particular it enumerates tuples with $\xi_i \neq \xi_j, i \neq j$. Therefore, for unordered n-tuples $\widetilde{\Xi}_n = \{\xi_1, \ldots, \xi_n | \xi_i \in \Xi\}$ we have $C(\widetilde{\Xi}_n) = C(\Xi)^n/n!$. Let $H = \bigoplus_{n=1}^{\infty} H_n$ be the gradation by degrees in Ξ. Then by (5), $C(H_n) = c_n C(\widetilde{\Xi}_n) = c_n C(\Xi)^n/n!$. Finally,

$$C(H) = \sum_{n=1}^{\infty} C(H_n) = \sum_{n=1}^{\infty} \frac{c_n}{n!} C(\Xi)^n = C(\mathbb{V}, C(\Xi)). \qquad \blacksquare$$

5. Complexity of the product of Lie varieties

Suppose that $\mathbb{M}$, $\mathbb{V}$ are varieties of Lie algebras. Then $\mathbb{M} \cdot \mathbb{V}$ is the class of all Lie algebras L such that there exists an ideal $H \subset L$ with $H \in \mathbb{M}$, $L/H \in \mathbb{V}$ [1]. For example, $\mathbb{V} = \mathbb{N}_{s_q} \ldots \mathbb{N}_{s_1}$ is the product $\mathbb{V} = \mathbb{N}_{s_q} \cdot \ldots \cdot \mathbb{N}_{s_2} \cdot \mathbb{N}_{s_1}$. If the base field is infinite, then all varieties of Lie algebras form a free semigroup with 0 and 1 with respect to this operation [1].

Our main goal is to prove a stronger version of the conjecture from [15], that the level of the product of two Lie varieties is equal to the sum of levels of summands.

THEOREM 5.1: *Let $\mathbb{M}$, $\mathbb{V}$ be varieties of Lie algebras and $\mathbb{M}$ is multihomogeneous. Suppose that* $\operatorname{Comp} \mathbb{V} = (q, \rho, \sigma)$, *where all these complexity numbers are nontrivial. Also, suppose that* $\operatorname{Comp}^1 \mathbb{M} = p$ *and* $\operatorname{Comp}^2 \mathbb{M} = \alpha > 0$. *Then*

(1)
$$\operatorname{Comp}(\mathbb{M} \cdot \mathbb{V}) \leq \begin{cases} (p+q, \rho, \sigma\alpha), & q = 1; \\ (p+q, \rho, \sigma), & q \geq 2. \end{cases}$$

(2) *In the first assertion we have equality if one of the conditions holds:*
   (a) $\operatorname{Comp} \mathbb{V} = \underline{\operatorname{Comp}} \mathbb{V} = (q, \rho, \sigma)$; *in particular,* $q = 1$.
   (b) $\underline{\operatorname{Comp}}^1 \mathbb{M} = p$.

(3) *If both conditions* (2a) *and* (2b) *are satisfied, then* $\underline{\operatorname{Comp}}(\mathbb{M} \cdot \mathbb{V}) = \operatorname{Comp}(\mathbb{M} \cdot \mathbb{V})$.

Proof: Let $L = F(\mathbb{M} \cdot \mathbb{V}, X)$, $X = \{x_i | i \in \mathbb{N}\}$. Consider auxiliary algebras $A = F(\mathbb{M}, Z)$, with free generators $Z = \{z_i | i \in \mathbb{N}\}$ and $B = F(\mathbb{V}, Y)$, with free generators $Y = \{y_i | i \in \mathbb{N}\}$. Now we can construct the wreath product $W = A \operatorname{wr}_{\mathbb{M}} B$ [1]. This is the Lie algebra with the following properties: as a vector space $W \cong \overline{A} \oplus B$, $\overline{A}$ is an ideal in W, and $\overline{A}$ is the free algebra in $\mathbb{M}$ with the following free generators:

$$\Xi = \{[z_j, u_{i_1}, u_{i_2}, \ldots, u_{i_n}] | u_{i_1} \le u_{i_2} \le \ldots \le u_{i_n},\ n \ge 0,\ z_j \in Z\}, \tag{6}$$

where u_{i_p} are elements of an ordered basis for B. The mapping $\Psi(x_i) = y_i + z_i$, $i \in \mathbb{N}$ can be extended to a monomorphism $\Psi\colon L \to W$ [1].

The algebra W is generated by Y and Z. We consider the double complexity functions with respect to these sets. Ordered elements in (6) correspond to the basis of $U(B)$ and, by Lemmas 4.1 and 4.2,

$$\mathcal{C}(\Xi, y, z) = z \cdot \mathcal{C}(U(B), y) = z \cdot \exp(\mathcal{C}(B, y)) = z \cdot \exp(\mathcal{C}(\mathbb{V}, y)).$$

We apply Lemma 4.3:

$$\mathcal{C}(W, y, z) = \mathcal{C}(\overline{A}) + \mathcal{C}(B) = \mathcal{C}(\mathbb{M}, z\exp(\mathcal{C}(\mathbb{V}, y))) + \mathcal{C}(\mathbb{V}, y).$$

Suppose that we have a double complexity function in which we identify variables

$$\begin{aligned} f(y, z) &= \sum_{n=1,m=1}^{\infty} \frac{c_{nm}}{n!m!} y^n z^m, \\ g(x) &= f(x, x) = \sum_{n=1}^{\infty} \frac{c_n}{n!} x^n; \end{aligned}$$

then

$$c_n = \sum_{a+b=n} \binom{n}{a} c_{ab}.$$

Let us consider the image of $P_n(\mathbb{M}\mathbb{V}, \{x_1, \ldots, x_n\})$ under $\Psi\colon L \to W$, $\Psi(x_i) = y_i + z_i$. In this image one has summands with different choices of y_i, z_i; in particular, there are $\binom{n}{a}$ ways to decide which a elements x_i will be substituted by y_i, while the others are being substituted by z_i. Then $c_{ab}(W)$ gives the number of elements in W with this fixed choice of y_i, z_i. This yields the bound

$$\mathcal{C}(\mathbb{M}\mathbb{V}, x) \prec \mathcal{C}(W, y, z)\Big|_{\substack{y=x \\ z=x}} = \mathcal{C}(\mathbb{M}, x\exp(\mathcal{C}(\mathbb{V}, x))) + \mathcal{C}(\mathbb{V}, x). \tag{7}$$

Now let us obtain a lower bound for $\mathcal{C}(\mathbb{MV})$. First, we consider $\widetilde{W} = W/\overline{A}^2$; then $\widetilde{W} \cong \overline{\overline{A}} \oplus B$, where $\overline{\overline{A}}$ is the abelian Lie algebra with the basis formed by images of Ξ. Also $\widetilde{W} \cong \widetilde{A}\,\mathrm{wr}_{\mathbb{A}}\, B$, where $\widetilde{A} = F(\mathbb{A}, Z)$; for simplicity we use the same letters Z, z_i also for $\widetilde{A}$ and $\overline{\overline{A}}$. Denote by $a \cdot b$, $a \in \overline{\overline{A}}$, $b \in U(B)$ the extension of the right action $a \cdot b = [a, b]$, $a \in \overline{\overline{A}}$, $b \in B$. In our case $\overline{\overline{A}} = \oplus_{z_i \in Z} z_i \cdot U(B)$ is the free $U(B)$-module [1]. Let $g(X_1, \ldots, X_s)$ be a multilinear identity of the minimal degree for $\mathbb{V}$; then the construction of the verbal ideal of the product [1] shows that it is not the identity for $\mathbb{AV}$. We remark that the image of $\widetilde{\Psi}\colon L \to \widetilde{W}$ is naturally isomorphic to $F(\mathbb{AV}, X)$, therefore

$$\widetilde{\Psi}(g(x_1, \ldots, x_s)) = f \neq 0,$$
$$f = \sum_{i=1}^{s} z_i \cdot f_i \in \overline{\overline{A}}, \quad f_i = f_i(y_1, \ldots, \widehat{y_i}, \ldots, y_s) \in U(B).$$

Then

$$\widetilde{\Psi}([g(x_1, \ldots, x_s), x_{i_{s+1}}, \ldots, x_{i_n}]) = f \cdot y_{i_{s+1}} \cdots y_{i_n},$$
$$\widetilde{\Psi}(P_n(L, \{x_1, \ldots, x_n\})) \supset f \cdot P_{n-s}(U(B), \{y_{s+1}, \ldots, y_n\}).$$

In the same way, for any $X' = \{x_{i_1}, \ldots, x_{i_n}\} \subset X$, $n \geq s$, we can fix s elements and $\widetilde{\Psi}(P_n(L, X')) \subset \overline{\overline{A}}$ contains $a_n = c_{n-s}(U(B))$ linearly independent elements. Thus, to any $X' \subset X$, $|X'| = n$, there correspond a_n linearly independent multilinear elements in $\overline{\overline{A}}$. Denote by Δ the set of all elements in $\overline{\overline{A}}$ which are obtained in this way. All these elements are obtained from $\widetilde{\Psi}(x_i)$, $x_i \in X$ and we consider the complexity function with respect to X,

$$\mathcal{C}(\Delta, x) = \sum_{n=s}^{\infty} \frac{a_n}{n!} x^n = \sum_{n=s}^{\infty} \frac{c_{n-s}(U(B))}{n!} x^n = \underbrace{\int_0^x \cdots \int_0^x}_{s \text{ times}} \mathcal{C}(U(B), x)\, dx \cdots dx. \tag{8}$$

Let us return to W. Any set in the free algebra $\overline{A} = F(\mathbb{M}, \Xi)$, which is linearly independent modulo its commutator subalgebra $\overline{A}^2$, generates a free subalgebra in the multihomogeneous variety $\mathbb{M}$ [1]. If we take pairwise distinct sets $X_1, \ldots, X_t$ then all respective elements in Δ are linearly independent, so Δ is the linearly independent set. Now (8) is the complexity function for the free generating set Δ of $F(\mathbb{M}, \Delta) \subset \overline{A}$. By Lemma 4.3 we obtain the lower estimate,

$$\begin{aligned} \mathcal{C}(\mathbb{MV}, x) &\succ \mathcal{C}(F(\mathbb{M}, \Delta), x) = \mathcal{C}(\mathbb{M}, \mathcal{C}(\Delta, x)) \\ &= \mathcal{C}\Big(\mathbb{M}, \underbrace{\int_0^x \cdots \int_0^x}_{s \text{ times}} \exp(\mathcal{C}(\mathbb{V}, x))\, dx \cdots dx\Big). \end{aligned} \tag{9}$$

Now we are in a position to prove the theorem. Our generating functions have nonnegative coefficients and we can consider that in (7) and (9) we have ordinary inequalities for all $x = r \in \mathbb{R}^+$. By hypothesis $\operatorname{ord}_{q-1} \mathcal{C}(\mathbb{V}, x) = \rho$, $\operatorname{typ}_{q-1} \mathcal{C}(\mathbb{V}, x) = \sigma$ and, after the exponents in (7) and (9), we obtain functions with the same order and type, but of the next level q. As above, the integrals in (9) do not change the level, order and type. Also, the factor x in (7) does not change the character of the growth. Let us denote the arguments of $\mathcal{C}(\mathbb{M}, *)$ in the right sides of (7) and (9) by $g(z)$ and $h(z)$, respectively. Then our considerations yield

$$g(r) \sim^{\sigma} \exp^{(q)}(\sigma r^{\rho}), \qquad h(r) \sim^{\sigma} \exp^{(q)}(\sigma r^{\rho}). \tag{10}$$

Let us prove assertion (1). According to (10), for any $\varepsilon > 0$ beginning with some number $g(r)$ is bounded by $\exp^{(q)}((\sigma+\varepsilon)r^{\rho})$; also $\mathcal{C}(\mathbb{M}, r) \leq \exp^{(p-1)}(r^{\alpha+\varepsilon})$ for sufficiently large r. After substitution, the right hand side of (7) is bounded asymptotically by $\exp^{(p+q-1)}((\sigma+2\varepsilon)r^{\rho})$ if $q \geq 2$ and $\exp^{(p)}((\sigma\alpha+2\varepsilon)r^{\rho})$ if $q = 1$. Since $\varepsilon > 0$ was taken arbitrarily, this proves the desired upper bound.

Let us consider (2a); here we are given $\underline{\operatorname{Comp}}\, \mathbb{V} = \operatorname{Comp} \mathbb{V} = (q, \rho, \sigma)$. Our assumption means that for any $\varepsilon > 0$ we have $\mathcal{C}(\mathbb{V}, r) \overset{\mathrm{a}}{\geq} \exp^{(q-1)}((\sigma-\varepsilon)r^{\rho})$. Now we apply Lemma 2.1 to construct an entire function with nonnegative coefficients $h_0(z)$ such that $\operatorname{ord}_q h_0(z) = \rho$, $\operatorname{typ}_q h_0(z) = \sigma - 2\varepsilon$. So, $\exp \mathcal{C}(\mathbb{V}, r) \overset{\mathrm{a}}{\geq} h_0(r)$; by subtracting some constant from the left side we can convert this inequality into an ordinary inequality. This inequality remains valid after taking s times the integral. By Lemma 2.1 we have $h(r) \overset{\mathrm{a}}{\geq} \exp^{(q)}((\sigma - 3\varepsilon)r^{\rho})$. Since $\varepsilon > 0$ was taken arbitrarily, and due to (10), we obtain $h(r) = \exp^{(q)}((\sigma + o(1))r^{\rho})$. It remains to recall that $\mathcal{C}(\mathbb{M}, r) \sim^{\alpha} \exp^{(p-1)}(r^{\alpha})$ and use these values in (9). If $q = 1$ then, by Theorem 3.5, $\underline{\operatorname{Comp}}\, \mathbb{V} = \operatorname{Comp} \mathbb{V}$.

Case (2b). By assumption, there exists $\alpha' > 0$ such that

$$\mathcal{C}(\mathbb{M}, r) \overset{\mathrm{a}}{\geq} \exp^{(p-1)}(r^{\alpha'}).$$

Due to (10), $h(r) \sim^{\sigma} \exp^{(q)}(\sigma r^{\rho})$ and these values we substitute into (9).

Assertion (3) is proved by combining the previous arguments. ∎

Corollary 5.1: *Under the hypothesis of the theorem*

$$\mathcal{C}\Big(\mathbb{M}, \underbrace{\int_0^x \dots \int_0^x}_{s \text{ times}} \exp(\mathcal{C}(\mathbb{V}, x))dx \dots dx\Big) \prec \mathcal{C}(\mathbb{M}\mathbb{V}, x)$$

$$\prec \mathcal{C}(\mathbb{M}, x\exp(\mathcal{C}(\mathbb{V}, x))) + \mathcal{C}(\mathbb{V}, x),$$

where s is the minimal degree of an identity in $\mathbb{V}$.

Remark: We cannot claim equality in the first assertion of Theorem 5.1 without additional assumptions. Just to explain this we observe that the upper limit of a composition of two functions does not always coincide with the composition of upper limits. Our additional assumptions say that one of the functions behaves "smoothly".

Remark: Theorem 5.1 yields by induction the upper bounds for Theorem 1.1. But we rely heavily on Theorems 2.1 and 2.2, which contain upper limits only, and we cannot derive the lower bounds in this way.

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